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Note

Infinite-Level Replicating Dissections of Plane Figures

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If a figure (here called the *replication*) is made up of k figures similar to it (here called the *replicators*), that figure is said to be *replicating of order* k, or *rep-*k. We deal in this paper only with cases in which the replicators are congruent.

One of the examples of replicating figures given by S. W. Golomb in [1] is the "snail" of Figure 1. We shall call this an "infinite-level dissection" and

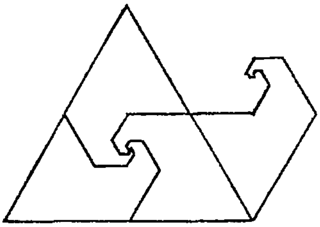

FIGURE 1

show how such dissections can be made. In our discussion of levels of dissection, the unused replicators at one level of dissection, if they have not been eliminated, become the replications to be dealt with in the next level.

The "snail" is a dissection of a rep-4 equilateral triangle, made using the three-fold polar symmetry of its replicating pattern (Figure 2). The pole

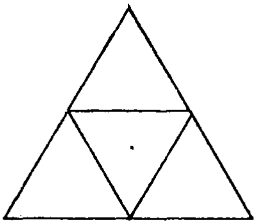

FIGURE 2

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lies within one of the replicators. In our procedure here, we retain one or more replicators at a level, and eliminate any which are symmetric with those retained. We cannot retain the replicator containing the pole, because doing do would require us to eliminate it too. Of the other three replicators we retain one and eliminate those symmetric with it from further consideration. (Figure 3) This completes the first level of our dissection.

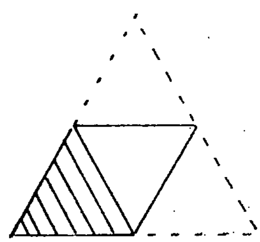

FIGURE 3

The first level replicator containing the pole was the only one neither retained nor eliminated at that level, so it is the only replication with which we must deal at the second level of dissection. The replicator retained at the second level must be in a similar position with respect to the second level replication as the replicator retained at the first level was with respect to its replication. We retain one such figure and eliminate those symmetric to it (Figure 4).

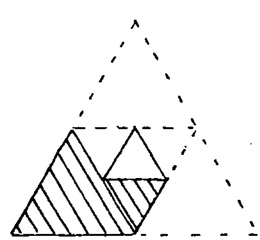

FIGURE 4

The second-level replicator we have retained is at a 60° rotation of the direction from the pole of the replicator retained at the first level. As can be seen, the union of the replicators retained at the $x + 1$ level and the x level must be similar to the union of the replicators retained at the first and second levels, so this 60° rotation must be maintained; however, the coincident axial symmetry of this particular figure allows for the direction of the angle of rotation to remain the same or be changed at the third level of dissection in this case.

If the direction of rotation remains the same at the third level, it must remain the same through all the levels of dissection, because the union of the replicators retained at the $x + 2$, $x + 1$, and x levels must be similar to the union of those retained at the first, second, and third levels. An infinite number of such levels of dissection would result in a "snail".

If the direction of rotation is changed at the third level, it must be changed at every level for this three-level similarity to remain, and an infinite number of levels in this case would result in another of Golomb's examples from [1], the "carpenter's plane", (Figure 5).

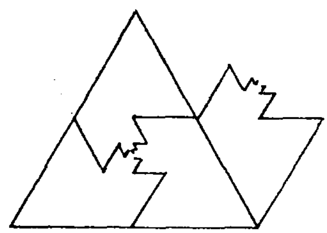

FIGURE 5

Now consider a rep-4 general parallelogram, which has a replicating pattern with two-fold polar symmetry (Figure 6). In this case, the pole of symmetry falls within none of the replicators, so it seems that either one or two may be retained if we eliminate any replicators symmetric with them.

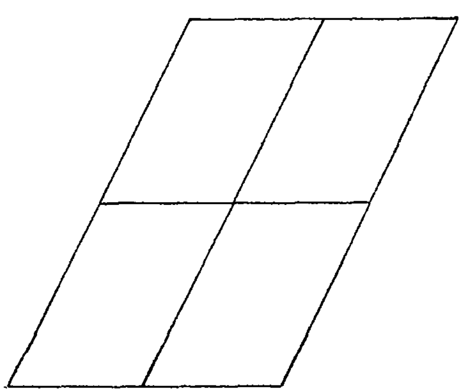

FIGURE 6

This is correct, but in the latter instance, no replicators remain for dissection at the second level. The figure will still replicate (the resulting figure in this case being just another parallelogram), but is what we will call a *one-level* *dissection*, and, as such, falls outside the bounds of our discussion here. One-level dissections are discussed both by Golomb in [1] and Grossman in [2].

We retain only one replicator at the first level (Figure 7) and are concerned with the dissection of two second level replications, each with its own pole,

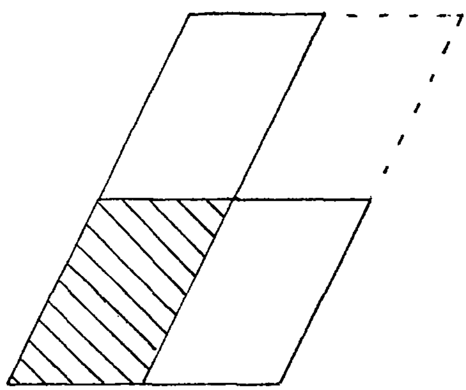

FIGURE 7

and each symmetric to the other with respect to the pole of the first level replication. Retention of one replicator in one of the second-level replications necessitates the elimination both of the replicator symmetric with it about the pole of its replication and of the replicator symmetric with it about the pole of the first level replication.

In this case, then, the replicators retained at the second level must be in the same direction from their respective poles. The two cases possible are shown in Figures 8 and 9. The pattern of retention and elimination is already determined for all levels when the second level dissection is made because the

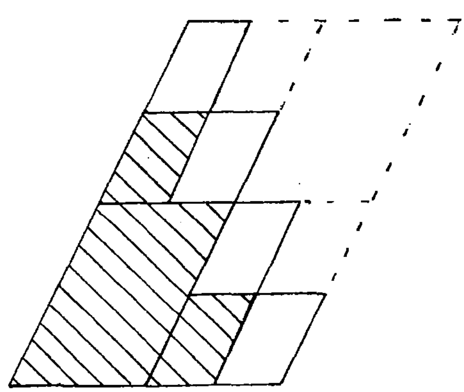

FIGURE 8

union of the replicators retained at the $x + 2$ level, the $x + 1$ level, and the x level within an x level replication must be similar to the union of the first-, second-, and third-level replictors retained within the first level replication. An infinite number of levels, then, if the first two are as in Figure 8, would result in a triangle. The first four levels of the figure the first two levels of which are shown in Figure 9, are shown in Figure 10.

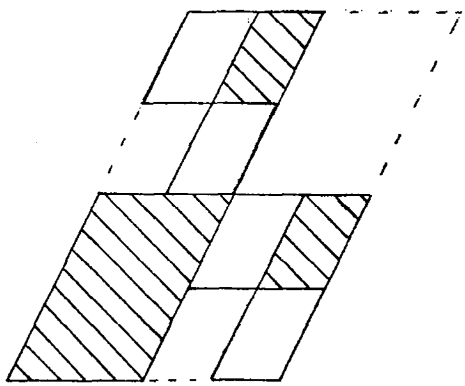

FIGURE 9

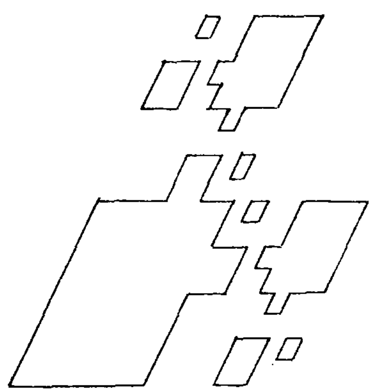

FIGURE 10

Analogous rules enable us to make replicating dissections from other figures whose replicating patterns have polar symmetry.

If a figure's replicating pattern has axial symmetry and the figure is rep-k, $k > 2$, that figure may be dissected into other rep-k figures. Again the union of the replicators retained at the $x + 2$ level, the $x + 1$ level, and the x level within an x-level replication must be similar to the union of the first-, second-, and third-level replicators retained within the first level replication.

For purposes of demonstrating such dissections, we will first use the example of Figure 11, shown by Golomb in [1]. The replicating pattern of the figure has axial symmetry, and we will demonstrate two of the four rep-4 dissections possible of this figure with respect to axial symmetry.

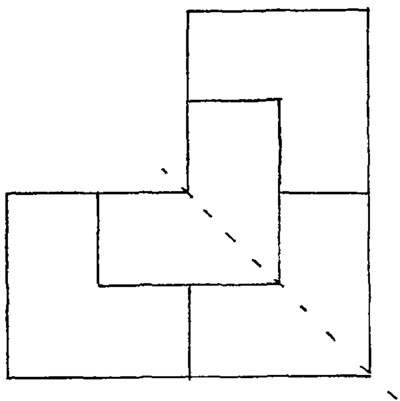

FIGURE 11

At the first level, we can retain neither of the two replicators through which the axis of symmetry passes, because doing so would necessitate eliminaring the same replicator. We retain one of the remaining figures and eliminate its reflection (Figure 12), leaving us two second level replications. The replications are not symmetric with each other, and so are independent replications in this second level. Of course, a replicator chosen for retention at the second level must be in a similar position with respect to its replication as either the first-level figure chosen for retention or its reflection with respect to the first level replication.

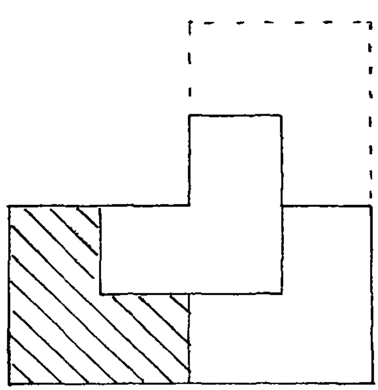

FIGURE 12

We choose the second-level retentions shown in Figure 13. These retentions, in this case, determine the replicators to be retained at all later levels. Therefore, four levels of such a dissection would be as shown in Figure 14.

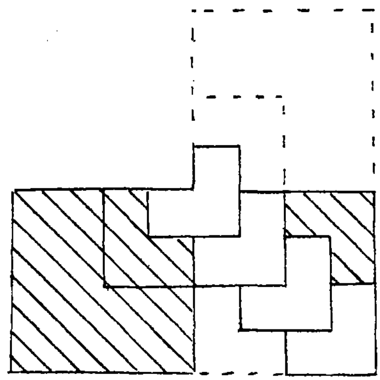

FIGURE 13

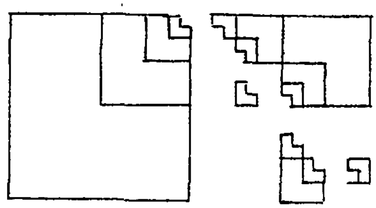

FIGURE 14

We choose for retention at the second level in another example the two replicators shown in Figure 15. An infinite number of levels in this instance results in the trapezoid shown in Figure 16, another of Golomb's examples from [1].

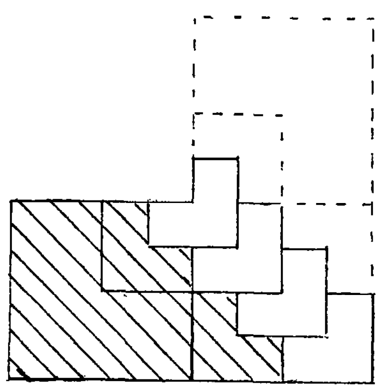

FIGURE 15

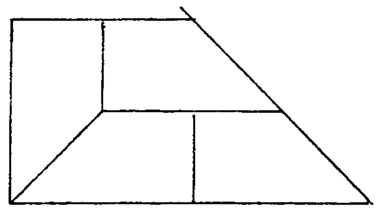

FIGURE 16

Another figure having a replicating pattern with axial symmetry is the rep-4 isosceles right triangle of Figure 17. The axis of symmetry doesn't pass through any of the replicators, so we can retain one or two at the first level. Retention of two results in a one-level dissection, so we will only consider cases where one is retained.

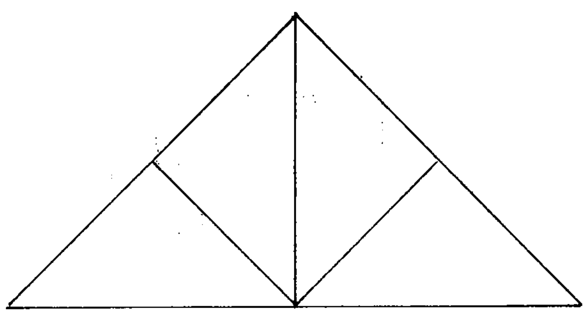

FIGURE 17

The first step could result in the retention of the replicator shown in Figure 18. In this case, the two replications in the second level are reflections of each other. Therefore, the replicators chosen at this level cause the elimination of their reflections both across the axis of their own replications and across

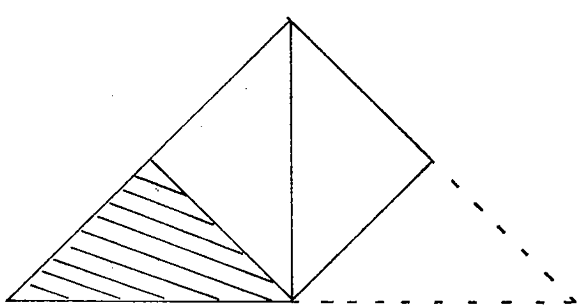

FIGURE 18

the axis of the first level replication. The first four levels of two possible dissections are shown in Figure 19.

If the first-level replication shown in Figure 20 is retained at the first level of dissection, an analogous situation is encountered. The second-level replications are again reflections of each other. The procedure is the same.

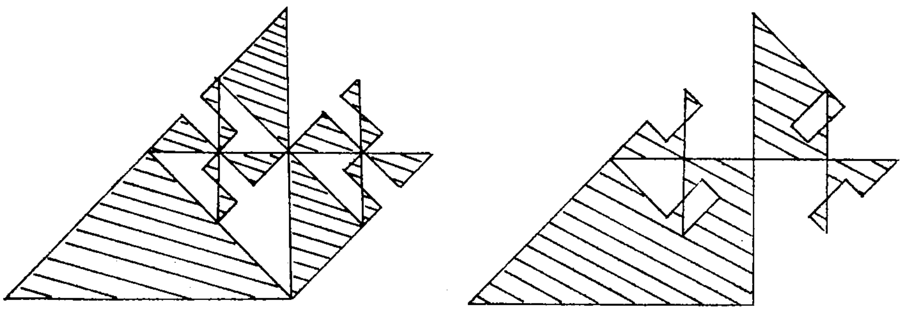

FIGURE 19

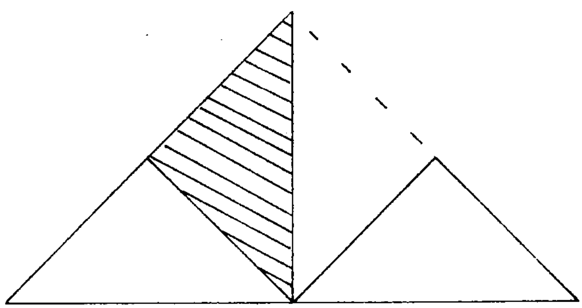

FIGURE 20

Analogous procedures can be followed to dissect other replicating figures of order k, $k > 2$, whenever the replicating pattern of such a figure has axial symmetry.

These are basic procedures. Variations are possible in some figures having biaxial, triaxial, or quadraxial symmetry, and also in figures having m-fold polar symmetry, $m > 2$.

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2. H. D. GROSSMAN, Fun with lattice points, *Scripta Math.* (1948), p. 157–159.